Multifractal analysis of images:
New connexions between analysis and geometry

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Abstract: Natural images can be modelled as patchworks of homogeneous textures with rough contours. The following stages play a key role in their analysis:

- Separation of each component
- Characterization of the corresponding textures
- Determination of the geometric properties of their contours.

Multifractal analysis proposes to classify functions by using as relevant parameters the dimensions of their sets of singularities. This framework can be used as a classification tool in the last two steps enumerated above. Several variants of multifractal analysis were introduced, depending on the notion of singularity which is used. We describe the variants based on Hölder and L^p regularity, and we apply these notions to the study of functions of bounded variation (indeed the BV setting is a standard functional assumption for modelling images, which is currently used in the first step for instance). We also develop a multifractal analysis adapted to contours, where the regularity exponent associated with points of the boundary is based on an accessibility condition. Its purpose is to supply classification tools for domains with fractal boundaries.

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1 Mathematical modelling of natural images

In order to develop powerful analysis and synthesis techniques in image processing, a prerequisite is to split the image into simpler components, and to develop some classification procedures for these components. Consider the example of a natural landscape: It consists in a superposition of different pieces which present some homogeneity. Perhaps, there will be a tree in the foreground, mountains in the background and clouds at the top of the picture. An efficient analysis procedure should first be able to separate these components which will display completely unrelated features and analyse each of them separately, since they will individually present some homogeneity. Considering an image as a superposition of overlapping components is referred to as the “dead-leave” model, introduced by Matheron, see [14] (and [2] for recent developments). Each piece appears as a relatively homogeneous part which is “cut” along a certain shape Ω. The homogeneous pieces are the “textures” and their modelling can be performed by interpreting them as the restriction on the “shape” Ω of random fields of $\mathbb{R}^2$ with particular statistical properties (stationarity,...). If a statistical model depending on a few parameters is picked, then one needs to devise a robust statistical test in order to estimate the values of these parameters. Afterwards, the test can be used as a classification tool for these textures. Furthermore, once the relevant values of the parameters have been identified, the model can be used for simulations. The procedure is now standard to classify and generate computer simulations of clouds for instance, see [1, 16].

Another problem is the modelling of the shape of Ω ; indeed natural scenes often do not present shapes with smooth edges (it is typically the case for the examples of trees, mountains or clouds that we mentioned) and the challenge here is to develop classification tools for domains with non-smooth (usually "fractal") boundaries. Until recently, the only mathematical tool available was the box-dimension of the boundary (see Definition 3.1) which is an important parameter but nevertheless very insufficient for classification (many shapes share the same box dimension for their boundary, but clearly display very different features).

Let us come back to the separation of the image into "simpler" components that present some homogeneity. It can be done using a "global" approach: The image is initially stored as grey-levels $f(x, y)$ and is approximated by a simple “cartoon” $u(x, y)$. What is meant by "simple" is that textures will be replaced by smooth pieces and rough boundaries by piecewise smooth curves[1]. The building of a mathematical model requires to summarize these qualitative assumptions by choosing an appropriate function space setting. In practice, the space BV (for "bounded variations") is usually chosen. A function f belongs to BV if its gradient (in the distribution sense) is a bounded measure (the name BV refers to the one-dimensional case where a function f belongs to BV if the sums $\sum_i |f(x_{i+1}) - f(x_i)|$ are uniformly bounded, no matter how

[1]This kind of simplification was anticipated by Hergé in his famous "Tintin" series and by his followers of the Belgian "la ligne claire" school.

we chose the finite increasing sequence x_i). Indeed, the space BV presents several of the required features : It allows only relatively smooth textures but, on the other hand, it allows for sharp discontinuities along smooth lines (or hypersurfaces, in dimension > 2). It is now known that natural images do not belong to BV, see [8], but this does not prevent BV to be used as a model for the "sketch" of an image: f is decomposed as a sum $u+v$ where $u \in BV$ and v is an approximation error (for instance it should have a small L^2 norm) and one uses a minimization algorithm in order to find u. In such approaches, we may expect that the discontinuities of the "cartoon" u will yield a first approximation of the splitting we were looking for. Such decompositions are referred to as "$u + v$" models and lead to minimization algorithms which are currently used; they were initiated by L. Rudin and S. Osher, and recent mathematical developments were performed by Y. Meyer (see [15] and references therein).

Once this splitting has been performed, one can consider the elementary components of the image (i.e. shapes that enclose homogenous textures) and try to understand their geometric properties in order to obtain classification tools ; at this point, no a priori assumption on the function is required; one tries to characterize the properties of the textures and of the boundaries by a collection of relevant mathematical parameters; these parameters should be effectively computable on real images in order to be used as classification parameters and hence for model selection. Multifractal analysis is used in this context: It proposes different pointwise regularity criteria as classification tools and it relates them to "global" quantities that are actually computable.

The different pointwise quantities (regularity exponents) which are used in multifractal analysis are exposed in Section 2, where we also recall their relationships.

In Section 3, we deal with "global" aspects : The tools (fractional dimensions) used in order to measure the sizes of sets with a given pointwise regularity exponent are defined (they are referred to as spectra of singularities). We also draw a bridge between these local analysis tools and the global segmentation approach described above :The implications the BV assumption on the multifractal analysis of a function are derived. The results of this section supply new tools in order to determine if particular images (or homogenous parts of images) belong to BV.

In Sections 4 and 5, we concentrate on the analysis of domains with fractal boundaries: Section 4 deals with general results concerning the pointwise exponents associated with these domains and Section 5 deals with their multifractal analysis. Apart from image processing, there are other motivations for the multifractal analysis of fractal boundaries, e.g. in physics and chemistry: Turbulent mixtures, aggregation processes, rough surfaces, see [12] and references therein.

2 Pointwise smoothness

Each variant of multifractal analysis is based on a definition of pointwise smoothness. In this section, we introduce the different definitions used, explain their motivations and their relationships.

2.1 Pointwise exponents for functions and measures

The most simple notion of smothness of a function is supplied by C^k differentiability. Recall that a bounded function f belongs to $C^1(\mathbb{R}^d)$ if it has everywhere partial derivatives $\frac{\partial f}{\partial x_i}$ which are continuous and bounded; C^k differentiability for $k \geq 2$ is defined by recursion: f belongs to $C^k(\mathbb{R}^d)$ if it belongs to $C^1(\mathbb{R}^d)$ and each of its partial derivatives $\frac{\partial f}{\partial x_i}$ belongs to $C^{k-1}(\mathbb{R}^d)$. Thus a definition is supplied for *uniform* smoothness when the regularity exponent k is an *integer*. Taylor's formula follows from the definition of C^k differentiability and states that, for any $x_0 \in \mathbb{R}^d$, there exists $C > 0$, $\delta > 0$ and a polynomial P_{x_0} of degree less than k such that

$$\text{if} \quad |x - x_0| \leq \delta, \qquad \text{then} \quad |f(x) - P_{x_0}(x)| \leq C|x - x_0|^k.$$

This consequence of C^k differentiability is just in the right form to yield a definition of *pointwise* smoothness which also makes sense for *fractional* orders of smoothness; following a usual process in mathematics, this result was turned into a definition.

Definition 2.1. *Let $\alpha \geq 0$, and $x_0 \in \mathbb{R}^d$; a function $f : \mathbb{R}^d \to \mathbb{R}$ is $C^\alpha(x_0)$ if there exists $C > 0$, $\delta > 0$ and a polynomial P_{x_0} of degree less than α such that*

$$\textit{if} \quad |x - x_0| \leq \delta, \qquad \textit{then} \quad |f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha. \tag{1}$$

The Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : \; f \;\; \text{is} \;\; C^\alpha(x_0)\}$.

Remarks: The polynomial P_{x_0} in (1) is unique and, if $\alpha > 0$, the constant term of P_{x_0} is $f(x_0)$; P is called the Taylor expansion of f at x_0 of order α; (1) implies that f is bounded in a neighbourhood of x_0; therefore, the Hölder exponent is defined only for locally bounded functions; it describes the local regularity variations of f. Some functions have a constant Hölder exponent: They display a "very regular irregularity". A typical example is the Brownian motion $B(t)$ which satisfies almost surely: $\forall x$, $h_B(x) = 1/2$.

Hölder regularity is the most widely used notion of pointwise regularity for functions. However, it suffers several drawbacks; one of them was patent at the very beginning of the introduction of multifractal analysis, in the mid-eighties; indeed, it was introduced as a tool to study the velocity of turbulent fluids, which is not necessarily a locally bounded function; and, as mentioned above, Hölder regularity can only be applied to locally bounded functions. Several mathematical drawbacks were already discovered at the beginning of the sixties by Calderón and Zygmund, see [4]. Another one which appeared recently is that the Hölder exponent of a function which has discontinuities cannot be deduced from the size of its wavelet coefficients. This is a very serious drawback for image analysis since images always contain objects partly hidden behind each other (this is referred to as the "occlusion phenomenon"), and therefore necessarily display discontinuities.

If B is a ball, let

$$\| f \|_{B,\infty}= \sup_{x\in B} |f(x)|, \quad \text{and, if } 1 \le p < \infty, \quad \| f \|_{B,p}= \left(\frac{1}{Vol(B)} \int_B |f(x)|^p dx \right)^{1/p};$$

finally let $B_r = \{x : \ |x - x_0| \le r\}$ (not mentioning x_0 in the notations won't introduce ambiguities afterwards). A clue to understand how the definition of pointwise Hölder regularity can be weakened (and therefore extended to a wider setting) is to notice that (1) can be rewritten $\| f - P_{x_0} \|_{B_r,\infty} \le Cr^\alpha$. Therefore, one obtains a weaker criterium by substituting in this definition the local L^∞ norm by a local L^p norm. The following definition was introduced by Calderón and Zygmund in 1961, see [4].

Definition 2.2. *Let $p \in [1, +\infty)$; a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ in L^p_{loc} belongs to $T^p_\alpha(x_0)$ if $\exists R, C > 0$ and a polynomial P_{x_0} of degree less than α such that*

$$\forall r \le R \quad \| f - P_{x_0} \|_{B_r,p} \le Cr^\alpha. \tag{2}$$

The p-exponent of f at x_0 is $h^p_f(x_0) = \sup\{\alpha : \ f \in T^p_\alpha(x_0)\}$.

It follows from the previous remarks that the Hölder exponent $h_f(x_0)$ coincides with $h^\infty_f(x_0)$. Note that (2) can be rewritten

$$\forall r \le R, \quad \int_{B_r} |f(x) - P_{x_0}(x)|^p dx \le Cr^{\alpha p+d}. \tag{3}$$

These p-smoothness conditions have several advantages when compared with the usual Hölder regularity conditions: They are defined as soon as f belongs locally to L^p and the p-exponent can be characterized by conditions bearing on the moduli of the wavelet coefficients of f, see [11]. Note that the $T^p_\alpha(x_0)$ condition gets weaker as p goes down, and therefore, for a given x_0, $p \mapsto h^p_f(x_0)$ is a decreasing function.

Let us now focus on the weakest possible case, i.e. when $p = 1$. First, recall that, if f is a locally integrable function, then x_0 is a *Lebesgue point* of f if

$$\frac{1}{Vol(B_r)} \int_{B_r} (f(x) - f(x_0))dx \longrightarrow 0 \quad \text{when} \quad r \longrightarrow 0. \tag{4}$$

Therefore, one can see the $T^1_\alpha(x_0)$ smoothness criterium as a way to quantify how fast convergence takes place in (4) when x_0 is a Lebesgue point of f.

Considering the L^1 norm of $f - P_{x_0}$ expresses an *average* smoothness: How close (in the mean) are f and a polynomial. Sometimes one rather wants to determine how *large* f is in the neighbourhood of x_0; then the relevant quantity is the rate of decay of the local L^1 norms $\int_{B(x_0,r)} |f(x)|dx$ when $r \to 0$. This quantity can also be considered for a nonnegative measure μ instead of an L^1 function: In that case, one considers $\int_{B(x_0,r)} d\mu = \mu(B(x_0, r))$. This leads us to the following pointwise *size* exponent.

Definition 2.3. *Let $p \in [1, +\infty)$; a nonnegative measure μ belongs to $S_\alpha(x_0)$ if there exist positive constants R and C such that*

$$\forall r \le R, \quad \int_{B_r} d\mu \le C r^\alpha. \tag{5}$$

The size-exponent of μ at x_0 is

$$s_\mu(x_0) = \sup\{\alpha : \ \mu \in S_\alpha(x_0)\} = \liminf_{r \to 0} \frac{\log \mu(B(x_0, r))}{\log r} \ .$$

If $f \in L^1$, then $s_f(x_0)$ is the size exponent of the measure $d\mu = |f(x)|dx$.

If $f \in L^1$ and if P_{x_0} in (3) vanishes, then the definitions of the 1-exponent and the size exponent of f coincide except for the normalization factor r^d in (3) which has been dropped in (5); thus, in this case, $s_f(x_0) = h^1_f(x_0) + d$. This discrepancy is due to historical reasons: Pointwise exponents for measures and for functions were introduced independently. It is however justified by the following remark which is a consequence of two facts: $\mu((x, y]) = |F(y) - F(x)|$, and the constant term of P_{x_0} is $F(x_0)$.

Remark: Let μ be a non-negative measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) < +\infty$ and let F be its repartition function defined by $F(x) = \mu((-\infty, x])$; if the polynomial P_{x_0} in (3) is constant, then $s_\mu(x_0) = h^1_F(x_0)$.

One does not substract a polynomial in the definition of the pointwise exponent of a measure because one is usually interested in the size of a measure near a point, not its smoothness: Consider the very important case where μ is the invariant measure of a dynamical system; then the size exponent expresses how often the dynamical system comes back close to x_0, whereas a smoothness index has no direct interpretation in terms of the underlying dynamical system.

We will need to use $T^p_\alpha(x_0)$ smoothness expressed in a slightly different form:

Proposition 2.4. *Let $f \in L^p_{loc}$, and $\alpha \in (0, 1]$; let $\overline{f}_r = \frac{1}{Vol(B_r)} \int_{B_r} f(x)dx$. Then*

$$f \in T^p_\alpha(x_0) \iff \left(\frac{1}{Vol(B_r)} \int_{B_r} \left| f(x) - \overline{f}_r \right|^p dx \right)^{1/p} \le C r^\alpha. \tag{6}$$

Proof. Suppose that $f \in T^p_\alpha(x_0)$ and let A be the constant polynomial which appears in the definition of T^p_α; then

$$\overline{f}_r - A = \frac{1}{Vol(B_r)} \int_{B_r} (f(x) - A)dx \ ;$$

Hölder's inequality yields that $|\overline{f}_r - A|$ is bounded by

$$\frac{1}{Vol(B_r)} \left[\int_{B_r} |f(x) - A|^p dx \right]^{1/p} (Vol(B_r))^{1/q} \le C(Vol(B_r))^{1/q - 1 + 1/p} r^\alpha = C r^\alpha \ .$$

Thus, $\overline{f}_r = A + O(r^\alpha)$. As a consequence, if we replace A by $\overline{f}_r$ in the quantity to be estimated in the definition of T^p_α, the error is $O(r^\alpha)$.

Conversly, suppose that (6) is true. Let r, r' such that $0 < r \leq r'$. We have

$$\|f - \overline{f}_r\|_{L^p(B_r)} \leq C r^{\alpha + d/p} \quad \text{and} \quad \|f - \overline{f}_{r'}\|_{L^p(B_{r'})} \leq C (r')^{\alpha + d/p} \ .$$

Since $r \leq r'$, $\|f - \overline{f}_{r'}\|_{L^p(B_r)} \leq C(r')^{\alpha + d/p}$; therefore

$$\|\overline{f}_{r'} - \overline{f}_r\|_{L^p(B_r)} \leq C(r')^{\alpha + d/p} \ ,$$

so that $|\overline{f}_{r'} - \overline{f}_r| \leq C(r')^\alpha$. It follows that $\overline{f}_r$ converges to a limit $\overline{f}_0$ when r goes to 0. Moreover $\overline{f}_r = \overline{f}_0 + O(r^\alpha)$ and therefore one can take $A = \overline{f}_0$.

2.2 Pointwise exponents for boundary points of domains

We will show how to draw distinctions between points of the boundary of a domain Ω, by associating to each of them an exponent, which may change from point to point along the boundary. This will allow us afterwards to perform a *multifractal analysis* of the boundary, i.e. to use as a discriminating parameter between different types of boundaries the whole collection of dimensions of the corresponding sets where this exponent takes a given value. Let us check if the exponents previously introduced can be used; the function naturally associated with a domain Ω is its characteristic function $1\!\!1_\Omega(x)$ which takes the value 1 on Ω and 0 outside Ω. The Hölder exponent of $1\!\!1_\Omega$ cannot play the role we expect, since it only takes two values: $+\infty$ outside $\partial\Omega$ and 0 on $\partial\Omega$. Let us now consider the p-exponents and the size exponent. We start by a toy-example: The domain $\Omega_\alpha \subset \mathbb{R}^2$ defined by

$$(x, y) \in \Omega_\alpha \quad \text{if and only if} \ \ |y| \leq |x|^\alpha.$$

At the point $(0, 0)$ one immediately checks that, if $\alpha \geq 1$, the p-exponent takes the value $(\alpha - 1)/p$, and the size exponent takes the value $\alpha + 1$. On the other hand, if $0 < \alpha < 1$, the p exponent takes the value $(1 - \alpha)/(\alpha p)$ but the size exponent is always equal to 2. This elementary computation shows the following facts: The p-exponent of a characteristic function can take any nonnegative value, the size exponent can take any value larger than 2; the 1-exponent and the size exponent give different types of information. The following proposition, whose proof is straightforward, gives a geometric interpretation for the size exponent of $1\!\!1_\Omega$.

Proposition 2.5. *Let Ω be a domain of $\mathbb{R}^d$ and let $x_0 \in \partial\Omega$; $1\!\!1_\Omega \in S_\alpha(x_0)$ if and only if $\exists R > 0$ and $C > 0$ such that $\forall r \leq R \ \ Vol(\Omega \cap B(x_0, r)) \leq C r^\alpha$.*

The following definition encapsulates this geometric notion.

Definition 2.6. *A point x_0 of the boundary of Ω is weak α-accessible if there exist $C > 0$ and $r_0 > 0$ such that $\forall r \leq r_0$,*

$$Vol\,(\Omega \cap B(x_0, r)) \leq C r^{\alpha+d}. \tag{7}$$

The supremum of all values of α such that (7) *holds is called the* weak accessibility exponent at x_0. *We denote it by $\alpha_w(x_0)$.*

Thus $\alpha_w(x_0)$ is a non negative number and is nothing but the size exponent of the measure $1\!\!1_\Omega(x)dx$ shifted by d. The following proposition of [12] shows that, for characteristic functions, all the p-exponents yield the same information and therefore one can keep only the 1-exponent.

Proposition 2.7. *Let Ω be a domain of $\mathbb{R}^d$ and let $x_0 \in \partial\Omega$; then $1\!\!1_\Omega \in T^p_\alpha(x_0)$ if and only if either $1\!\!1_\Omega \in S_{\alpha/p}(x_0)$ or $1\!\!1_{\Omega^c} \in S_{\alpha/p}(x_0)$, where Ω^c denote the complement of Ω.*

Following the same idea as above, one can also define a *bilateral* accessibility exponent of a domain which is the geometric formulation of the 1-exponent of the function $1\!\!1_\Omega$, see [12].

Definition 2.8. *A point x_0 of the boundary of Ω is bilaterally weak α-accessible if there exist $C > 0$ and $r_0 > 0$ such that $\forall r \leq r_0$,*

$$\min\Big[Vol\,(\Omega \cap B(x_0, r))\ \ ,\ Vol\,(\Omega^c \cap B(x_0, r))\Big] \leq C r^{\alpha+d}. \tag{8}$$

The supremum of all values of α such that (8) *holds is called the* bilateral weak accessibility exponent at x_0. *We denote it by $\beta_w(x_0)$.*

Remark 1: It follows immediately from the above definitions that the bilateral exponent $\beta_w(x_0)$ is the supremum of the unilateral exponents $\alpha_w(x_0)$ associated with Ω and Ω^c. In practice, using unilateral or bilateral exponents as classification tools in multifracal analysis will be irrelevant when Ω and Ω^c have the same statistical properties. It is the case when they are obtained by a procedure which makes them play the same role (for instance if $\partial\Omega$ is the edge of the fracture of a metallic plate). On the other hand, unilateral exponents should yield different types of information when the roles played by Ω and its complement are very dissymetric (electrodeposition aggregates for instance).

Remark 2: If $\Omega \in BV$, then, by definition $\text{grad}(1\!\!1_\Omega)$ is a measure, and therefore one could also consider an additional exponent, which is the size exponent of $|\text{grad}(1\!\!1_\Omega)|$. We won't follow this idea because, in applications, one has no direct access to the measure $\text{grad}(1\!\!1_\Omega)$, and we want to base our analysis only on information directly available from Ω.

We will also use the following alternative accessibility exponents.

Definition 2.9. *A point x_0 of the boundary of Ω is strong α-accessible if there exist $C > 0$ and $r_0 > 0$ such that $\forall r \le r_0$,*

$$Vol\,(\Omega \cap B(x_0, r)) \ge C r^{\alpha+d}. \tag{9}$$

The infimum of all values of α such that (9) *holds is called the* strong accessibility exponent at x_0. *We denote it by $\alpha_s(x_0)$.*

A point x_0 of the boundary of Ω is bilaterally strong α-accessible if there exist $C > 0$ and $r_0 > 0$ such that $\forall r \le r_0$,

$$\min\left[Vol\,(\Omega \cap B(x_0, r)) \ \ , \ Vol\,(\Omega^c \cap B(x_0, r))\right] \ge C r^{\alpha+d}. \tag{10}$$

The infimum of all values of α such that (10) *holds is called the* bilateral strong accessibility exponent at x_0. *We denote it by $\beta_s(x_0)$.*

The following result yields alternative definitions of these exponents.

Proposition 2.10. *Let $x \in \partial\Omega^{ess}$; then*

$$\alpha_w(x) + d = \liminf_{r \to 0} \frac{\log Vol\,(\Omega \cap B(x, r))}{\log r} \ , \ \alpha_s(x) + d = \limsup_{r \to 0} \frac{\log Vol\,(\Omega \cap B(x, r))}{\log r}.$$

Similar relations hold for the indices $\beta_w(x)$ and $\beta_s(x)$.

Other exponents associated with boundaries have been introduced; they were based on the notion of *density*, which we now recall.

Definition 2.11. *Let $x_0 \in \Omega$; the density of Ω at x_0 is*

$$D(\Omega, x_0) = \lim_{r \to 0} \frac{Vol(B(x_0, r) \cap \Omega)}{Vol(B(x_0, r))} \tag{11}$$

This limit does not necessarily exist everywhere; thus, if one wants to obtain an exponent which allows a classification of all points of $\partial\Omega$, the *upper density exponent* $\overline{D}(\Omega, x_0)$ or the *lower density exponent* $\underline{D}(\Omega, x_0)$ should rather be used; they are obtained by taking in (11) respectively a lim sup or a lim inf. The set of points where $D(\Omega, x_0)$ differs from 0 and 1 is called the *measure theoretic boundary*, see Chap. 5 of [22]. This allows to introduce topological notions which have a measure-theoretic content: The *measure theoretic interior* of Ω is the set of points satisfying $D(\Omega, x_0) = 1$; the *measure theoretic exterior* is the set of points satisfying $D(\Omega, x_0) = 0$, see Chap. 5 of [22] for more on these notions which bear some similarities with the ones we will develop in Section 4.1. Note that points with a positive weak-accessibility exponent all have a vanishing density, so that density exponents are a way to draw a distinction between different points of weak-accessibility 0. This refinement has been pushed even further when Ω has a finite perimeter (i.e. when $1\!\!1_\Omega \in BV$): Points of density 1/2 can be classified by considering points where the boundary is "close" to an hyperplane (see [22] for precise definitions); such points constitute the "reduced boundary" introduced by de Giorgi. We will come back to these classifications in Section 4.2.

3 Fractional dimensions, spectra and multifractal analysis

3.1 Fractional dimensions

In order to introduce global parameters which allow to describe the "fractality" of the boundary of a domain, we need to recall the notions of dimensions that will be used. Their purpose is to supply a classification among sets of vanishing Lebesgue measure in $\mathbb{R}^d$.

The simplest notion of dimension of a set E (and the only one that is computable in practice) is the upper box-dimension. It can be obtained by estimating the number of dyadic cubes that intersect E. Recall that a *dyadic cube* of scale j is of the form

$$\lambda = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_d}{2^j}, \frac{k_d+1}{2^j}\right), \quad \text{where } k = (k_1, \ldots k_d) \in \mathbb{Z}^d;$$

$\mathcal{F}_j$ denotes the set of dyadic cubes of scale j.

Definition 3.1. (Upper box-dimension) *Let E be a bounded set in $\mathbb{R}^d$ and $N_j(E)$ be the number of cube $\lambda \in \mathcal{F}_j$ that intersect E. The upper box-dimension of the set E is defined by*

$$\Delta(E) = \limsup_{j \to +\infty} \frac{\log(N_j(E))}{\log(2^j)} \ .$$

This notion of dimension presents two important drawbacks. The first one is that it takes the same value for a set and its closure. For example, the upper box-dimension of the set $\mathbb{Q}$ of rational numbers is equal to 1, but we would expect the dimension of a countable set to vanish. The second one is that it is not a σ-stable index, i.e. the dimension of a countable union of sets usually differs from the supremum of the dimensions of the sets. In order to correct these drawbacks, a very clever idea, introduced by C. Tricot in [21], consists in "forcing" the σ-stability as follows:

Definition 3.2. (Packing dimension) *Let $E \subset \mathbb{R}^d$; the packing dimension of E is*

$$\dim_P(E) = \inf \left(\sup_{i \in \mathbb{N}} [\Delta(E_i)] \ ; \ E \subset \bigcup_{i \in \mathbb{N}} E_i \right) \ ,$$

where the infimum is taken on all possible "splittings" of E into a countable union.

The Hausdorff dimension is the most widely used by mathematicians.

Definition 3.3. (Hausdorff dimension) *Let $E \subset \mathbb{R}^d$ and $\alpha > 0$. Let us introduce the following quantities : Let $n \in \mathbb{N}$; if $\Lambda = \{\lambda_i\}_{\ i \in \mathbb{N}}$ is a countable collection of dyadic cubes of scales at least n which forms a covering of E, then let*

$$\mathcal{H}^\alpha_n(E, \Lambda) = \sum_{i \in \mathbb{N}} \operatorname{diam}(\lambda_i)^\alpha, \quad \textit{and} \quad \mathcal{H}^\alpha_n(E) = \inf \left(\mathcal{H}^\alpha_n(E, \Lambda) \right) \ ,$$

where the infimum is taken on all possible coverings of E by dyadic cubes of scales at least n. The α-dimensional Hausdorff measure of E is

$$\mathcal{H}^{\alpha}(E) = \lim_{n\to+\infty} \mathcal{H}^{\alpha}_n(E).$$

The Hausdorff dimension of E is

$$\dim_H(E) = \sup\left(\alpha > 0 \ ; \ \mathcal{H}^{\alpha}(E) = +\infty\right) = \inf\left(\alpha > 0 \ ; \ \mathcal{H}^{\alpha}(E) = 0\right) \ .$$

Remark 1. Hausdorff measures extend to fractional values of d the notion of d-dimensional Lebesgue measure, indeed, $\mathcal{H}^d$ is the Lebesgue measure in $\mathbb{R}^d$. The Hausdorff dimension is an increasing σ-stable index.

Remark 2. The following inequalities are always true, see [6].

$$0 \le \dim_H(E) \le \dim_P(E) \le \Delta(E) \le d \ .$$

3.2 Spectra of singularities

In all situations described in Section 2, a "pointwise smoothness function" is associated to a given signal (this may be for example the Hölder exponent, the p-exponent or the size exponent). In the case where the signal is irregular, it is of course impossible to describe this function point by point. That is why one tries to obtain a statistical description, by determining only the dimensions of the sets of points with a given exponent. This collection of dimensions, indexed by the smoothness parameter is called the *spectrum of singularities*. Actually, two kinds of spectra are used, depending whether one picks the Hausdorff or the packing dimension, see Theorems 5.3 and 5.4 for estimates on such spectra. In the next section, we will estimate the p-spectrum of BV functions. This p-spectrum $d^p_f(H)$ is the Hausdorff dimension of the set of points whose p-exponent is H. If $p = \infty$, $d^{\infty}_f(H)$ is simply denoted by $d_f(H)$: It denotes the Hausdorff dimensions of the sets of points where the Hölder exponent is H, and is called the spectrum of singularities of f.

3.3 Multifractal analysis of BV functions

We saw that the space BV is currently used in order to provide a simple functional setting for "sketchy" images, i.e. images which consist of piecewise smooth pieces separated by lines of discontinuities which are piecewise smooth. This approach is orthogonal to the multifractal point of view; indeed, multifractal analysis makes no a priori assumption on the function considered and, therefore, is relevant also in the analysis of non smooth textures and irregular edges. In order to go beyond this remark, it is important to understand the implications of the BV assumption on the multifractal analysis of a function. They strongly depend on the number of variables of f ; therefore, though our main concern deals with functions defined on $\mathbb{R}^2$, considering the general case of functions defined on $\mathbb{R}^d$ will explain some phenomena which, if dealt with only for $d = 1$ or 2, might appear as strange numerical coincidences.

We start by recalling the alternative definitions of the space $BV(\mathbb{R}^d)$. Let Ω be an open subset of $\mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d)$. By definition,

$$\int_\Omega |Df| = \sup \left\{ \int_\Omega f \operatorname{div} g, \; g = (g_1, \cdots, g_d) \in C_0^1(\Omega, \mathbb{R}^d) \quad \text{and } \|g\|_\infty \leq 1 \right\} ,$$

where $\operatorname{div} g = \sum_{i=1}^d \frac{\partial g_i}{\partial x_i}$. This notation is justified as follows : An integration by parts shows that if $f \in C^1(\mathbb{R}^d)$, $\int_\Omega |Df| = \int_\Omega |\text{grad} f| dx$ where $\text{grad} f = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d})$.

Definition 3.4. *Let $\Omega \subset \mathbb{R}^d$, and $f \in L^1(\mathbb{R}^d)$; f belongs to $BV(\Omega)$ if $\int_\Omega |Df| < +\infty$.*

Recall that the alternative definition is: $f \in BV(\Omega)$ if $f \in L^1(\Omega)$ and $\text{grad}\, f$ (defined in the sense of distributions) is a Radon vector-measure of finite mass.

What is the correct setting in order to perform the multifractal analysis of a BV function ? In dimension 1, the alternative definition in terms of Radon measures immediately shows that a BV function is bounded (indeed a Radon measure is the difference of two positive measures and the primitive of a positive measure of finite mass is necessarily bounded). Therefore, one can expect that the BV assumption has a consequence on the "usual" spectrum $d_f(H)$ based on the Hölder exponent. On the other hand, if $d > 1$, then a BV function needs not be locally bounded (consider for instance the function $\frac{1}{\|x\|^\alpha}$ in a neighborhood of 0, for α small enough). A simple superposition argument shows that it may even be nowhere locally bounded ; therefore, we cannot expect the BV assumption to yield any information concerning the "usual" spectrum of singularities in dimension 2 or more. The following Sobolev embeddings precisely determine for which values of p a BV function locally belongs to L^p (see [7]).

Proposition 3.5. ([7]) *Let $d^\star = \frac{d}{d-1}$ ($d^\star$ is the conjugate exponent of d). If $f \in BV(\mathbb{R}^d)$ then*

$$\|f\|_{d^\star} \leq C(d) \int |Df| \; . \tag{12}$$

Moreover, if $B = B(x_0, r)$ and $\overline{f}_B = \frac{1}{Vol(B)} \int_B f(x)dx$,

$$\|f - \overline{f}_B\|_{L^{d^\star}(B)} \leq C(d) \int_B |Df| \; . \tag{13}$$

Since (12) states that $BV(\mathbb{R}^d)$ is embedded in $L^{d^\star}(\mathbb{R}^d)$, we can infer from this proposition that the "right" value of p in order to study the pointwise smooothness of functions in $BV(\mathbb{R}^d)$ is $p = d^\star$. The following result actually gives estimates of the $d^\star$-spectrum of BV functions.

Theorem 3.6. *Let $f \in BV(\mathbb{R}^d)$. The $d^\star$-spectrum of f satisfies*

$$d_f^{d^\star}(H) \leq H + (d-1) \ .$$

Proof of Theorem 3.6. If $d = 1$, f is the difference of two increasing functions. The theorem is a consequence of the classical bound $d(H) \leq H$ for probability measures, see [3] and the remark that, if $H \leq 1$, the size exponent of a positive measure and the Hölder exponent of its primitive coincide. We can therefore assume that $d \geq 2$.

We can clearly suppose that $H \leq 1$. Let us consider f on the unit cube $[0,1]^d$ and let $j \geq 0$. We split this cube into 2^{dj} dyadic cubes of width 2^{-j}. If λ is a dyadic cube in $\mathcal{F}_j$, let $TV(\lambda)$ denote the total variation of f on the ball $B_\lambda = B(\mu_\lambda, \sqrt{d}2^{-j})$ where μ_λ is the center of λ, i.e. $TV(\lambda) = \int_{B_\lambda} |Df|$. Let $\delta > 0$ and denote by $A(\delta, j)$ the set of λ's such that $TV(\lambda) \geq 2^{-\delta j}$ and by $N(\delta, j)$ its cardinal. Since only a finite number $\tilde{C}(d)$ of balls B_λ overlap,

$$N(\delta, j) 2^{-\delta j} \leq \sum_{\lambda \in A(\delta, j)} TV(\lambda) \leq C(d) \int |Df| \ .$$

Therefore

$$N(\delta, j) \leq C 2^{\delta j}. \tag{14}$$

Let x_0 be such that it only belongs to a finite number of $A(\delta, j)$. Let $\lambda_j(x_0)$ denote the dyadic cube of width 2^{-j} which contains x_0. For j large enough, $TV(\lambda_j(x_0)) \leq 2^{-\delta j}$. If $B = B(x_0, \sqrt{d}2^{-(j+1)})$, (13) implies that

$$\|f - \overline{f}_B\|_{L^{d^\star}(B)} \leq C \int_B |Df| \leq C \int_{B_\lambda} |Df| \leq C 2^{-\delta j};$$

thus, using Proposition 2.4, $f \in T^{d^\star}_{\delta - d/d^\star}(x_0)$ $(= T^{d^\star}_{\delta - d+1}(x_0))$. Denote

$$A_\delta = \limsup_{j \to +\infty} A(\delta, j) \ .$$

The set A_δ consists of point that belong to an infinite number of sets $A(\delta, j)$. Then, (14) implies that $\dim_H (A_\delta) \leq \delta$. If $x_0 \notin A_\delta$, we just showed that $f \in T^{d^\star}_{\delta - d+1}(x_0)$. It follows that the set of points of $d^\star$-exponent $\delta - d + 1$ is of Hausdorff dimension at most δ. In other words, $d_f^{d^\star}(\delta - d + 1) \leq \delta$, hence Theorem 3.6 holds.

Remark: Let us pick $\delta > d - 1$ but arbitrarily close to $d - 1$. We saw that A_δ has dimension less than δ and if $x_0 \notin A_\delta$, then x_0 belongs to $T^{d^\star}_\alpha$ for an $\alpha > 0$ so that x_0 is a Lebesgue point of f. It follows that, if f is a BV function, then the set of points which are not Lebesgue points of f has Hausdorff dimension at most $d - 1$. Related results are proved in Section 5.9 of [5] (see in particular Theorem 3).

Theorem 3.6 only gives an information on the $d^\star$-exponent and cannot give additional information on q-regularity for $q > d^\star$ since a function of $BV(\mathbb{R}^d)$ may nowhere be locally in L^q for such values of q. However, images are just grey-levels at each pixel and therefore ae encoded by functions that take values between 0 and 1. Therefore, a more realistic modelling is supplied by the assumption $f \in BV \cap L^\infty$. Let us now see if this additional assumption allows us to derive an estimate on the q-spectrum.

Lemma 3.7. *Let $f \in T^p_\alpha(x_0) \cap L^\infty(\mathbb{R}^d)$ for some $p \geq 1$ and let q satisfy $p < q < +\infty$. Then $f \in T^q_{\alpha p/q}(x_0)$.*

Proof. By assumption, $\|f - \overline{f}_{B_r}\|_{L^p(B_r)} \leq C r^{\alpha + d/p}$, where B_r denotes the ball $B(x_0, r)$. Let $\omega = \frac{p}{q}$, so that $0 < \omega < 1$; since f is bounded, by interpolation,

$$\|f - \overline{f}_{B_r}\|_{L^q(B_r)} \leq (2\|f\|_\infty)^{(1-\omega)} \|f - \overline{f}_{B_r}\|^\omega_{L^p(B_r)}.$$

Therefore, if $\beta = \alpha p/q$, then $\|f - \overline{f}_{B_r}\|_{L^q(B_r)} \leq C r^{(\alpha + d/p)\omega} = C r^{\beta + d/q}$.

Corollary 3.8. *Let $f \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $q \geq d^\star$. The q-spectrum of f satisfies*

$$d^q_f(H) \leq \frac{q}{d^\star} H + (d-1) \ .$$

Remark: Of course this inequality is relevant only when $H \leq \frac{d^\star}{q}$.

Proof. We come back to the proof of Theorem 3.6. We proved that outside the set A_δ, f belongs to $T^{d^\star}_{\delta - d + 1}(x_0)$. It follows from the previous lemma that f also belongs to $T^q_\gamma(x_0)$ for $\gamma = \left(\delta - \frac{d}{d^\star}\right) \frac{d^\star}{q} = \frac{\delta d^\star}{q} - \frac{d}{q}$. Since A_δ is of dimension at most δ, the corollary follows just as the end of Theorem 3.6.

4 Topological and geometric properties of the essential boundary

4.1 Essential boundary and modified domain

The geometric quantities introduced in Section 2 do not change if Ω is replaced by another set $\tilde{\Omega}$, as long as they differ by a set of measure 0. This is clear when we consider the function $1\!\!1_\Omega$ (viewed as a L^p_{loc}-function), the measure $1\!\!1_\Omega(x)dx$ or the indices α_w, α_s, β_w and β_s. Therefore, the only points of the boundary that are pertinent to analyse from a "measure" point of view are those for which $Vol(B(x_0, r) \cap \Omega) > 0$ and $Vol(B(x_0, r) \cap \Omega^c) > 0$. This motivates the following definition.

Definition 4.1. (Essential boundary) *Let Ω be a Borel subset of $\mathbb{R}^d$. Denote by $\partial\Omega^{ess}$ the set of points $x_0 \in \mathbb{R}^d$ such that for every $r > 0$,*

$$Vol(B(x_0, r) \cap \Omega) > 0 \quad \text{and} \quad Vol(B(x_0, r) \cap \Omega^c) > 0 \ .$$

The set $\partial\Omega^{ess}$ is called the essential boundary of Ω.

It is clear that $\partial\Omega^{ess} \subset \partial\Omega$. More precisely, we have the following characterization of $\partial\Omega^{ess}$; recall that, if A and B are subsets of $\mathbb{R}^d$, then $A\Delta B = (A \cup B) \setminus (A \cap B)$.

Proposition 4.2. *Let $x \in \mathbb{R}^d$. Then, $x \in \partial\Omega^{ess}$ if and only if x is a boundary point of every Borel set Ω' such that $Vol(\Omega\Delta\Omega') = 0$.*

Remark: In particular, $\partial\Omega^{ess}$ is a closed subset of $\mathbb{R}^d$.

Proof of Proposition 4.2. Let

$$A = \bigcap_{Vol(\Omega\Delta\Omega')=0} \partial\Omega' \ .$$

It is clear that $\partial\Omega^{ess} \subset A$. Conversly, suppose for example that there exists $r > 0$ such that $Vol(\Omega \cap B(x, r)) = 0$. Define Ω' by $\Omega' = \Omega \setminus B(x, r)$. Then $Vol(\Omega\Delta\Omega') = 0$ and $x \notin \partial\Omega'$. $\bullet$

The essential boundary can also be defined as the support of the distribution $\mathrm{grad}(\mathbb{1}_\Omega)$. According to Proposition 4.2, it is natural to ask if there exists a modified Borel set $\tilde{\Omega}$ which is minimal in the sense that $Vol(\Omega\Delta\tilde{\Omega}) = 0$ and $\partial\Omega^{ess} = \partial\tilde{\Omega}$.

Proposition 4.3. (Modified domain) *Let Ω be a Borel set in $\mathbb{R}^d$. There exists a Borel set $\tilde{\Omega}$ such that*

$$Vol(\Omega\Delta\tilde{\Omega}) = 0 \quad \text{and} \quad \partial\Omega^{ess} = \partial\tilde{\Omega} \ .$$

In particular $\partial\tilde{\Omega} \subset \partial\Omega'$ for every Ω' such that $Vol(\Omega\Delta\Omega') = 0$. The Borel set $\tilde{\Omega}$ is called the modified domain of Ω.

Remarks: This notion is implicit in many books of geometric measure theory, see for instance [7] page 42. We can suppose in the following that $\Omega = \tilde{\Omega}$ and $\partial\Omega^{ess} = \partial\Omega$.

Proof of Proposition 4.3. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of of open balls which is a base for the usual topology in $\mathbb{R}^d$. Let

$$I^- = \{n \in \mathbb{N}\ ;\ Vol(B_n \cap \Omega) = 0\} \quad \text{and} \quad I^+ = \{n \in \mathbb{N}\ ;\ Vol(B_n \cap \Omega^c) = 0\} \ .$$

Observe that if $p \in I^-$ and $q \in I^+$, then, $B_p \cap B_q = \emptyset$. Define

$$\tilde{\Omega} = \left(\Omega \setminus \bigcup_{n\in I^-} B_n\right) \bigcup \left(\bigcup_{n\in I^+} B_n\right) \ .$$

It is clear that $Vol(\Omega\Delta\tilde{\Omega}) = 0$. There remains to prove that $\partial\tilde{\Omega} \subset \partial\Omega^{ess}$.

Let $x \in \partial\tilde{\Omega}$ and $r > 0$. Let n be such that $x \in B_n \subset B(x, r)$. Since x is in the closure of $\tilde{\Omega}$, $B_n \cap \tilde{\Omega} \neq \emptyset$. So, $n \notin I^-$ and $Vol(B_n \cap \Omega) > 0$. In the same way, x is in the closure of $\tilde{\Omega}^c$ and $B_n \cap \tilde{\Omega}^c \neq \emptyset$; thus $n \notin I^+$ and $Vol(B_n \cap \Omega^c) > 0$. Finally $Vol(B(x, r) \cap \Omega) > 0$ and $Vol(B(x, r) \cap \Omega^c) > 0$ so that $x \in \partial\Omega^{ess}$.

We can also define the *essential interior* and *essential closure* of Ω by

$$\overset{\circ}{\Omega}^{ess} = \Big\{x \in \mathbb{R}^d \ ; \ \exists r > 0 \ ; \ Vol(B(x,r) \cap \Omega^c) = 0\Big\}$$

and

$$\overline{\Omega}^{ess} = \Big\{x \in \mathbb{R}^d \ ; \ \forall r > 0, \ Vol(B(x,r) \cap \Omega) > 0\Big\} \ .$$

They are respectively open and closed subsets of $\mathbb{R}^d$ and satisfy $\partial\Omega^{ess} = \overline{\Omega}^{ess} \setminus \overset{\circ}{\Omega}^{ess}$.

4.2 Balanced points

We now explore the topological properties of the sets of points of the essential boundary $\partial\Omega^{ess}$ for which either β_w or β_s vanishes. We begin with a definition which identifies natural subsets of the sets of points with accessibility 0.

Definition 4.4. *Let $\Omega \subset \mathbb{R}^d$ be a Borel set and $x_0 \in \partial\Omega^{ess}$.*

1. *A point x_0 is strongly balanced if there exists $0 < \eta < 1/2$ and $r_0 > 0$ such that*

$$\forall r \leq r_0, \quad \eta \leq \frac{Vol(B(x_0,r) \cap \Omega)}{Vol(B(x_0,r))} \leq 1 - \eta \ .$$

2. *A point x_0 is weakly balanced if there exists $0 < \eta < 1/2$ such that*

$$\forall r_0 > 0, \ \exists r \leq r_0; \quad \eta \leq \frac{Vol(B(x_0,r) \cap \Omega)}{Vol(B(x_0,r))} \leq 1 - \eta \ .$$

We denote by $SB(\Omega)$ (resp. $WB(\Omega)$) the set of strongly (resp. weakly) balanced points in $\partial\Omega^{ess}$. It is clear that

$$SB(\Omega) \subset \{x_0 \in \partial\Omega^{ess} \ ; \ \beta_s(x_0) = 0\} \quad \text{and} \quad WB(\Omega) \subset \{x_0 \in \partial\Omega^{ess} \ ; \ \beta_w(x_0) = 0\} \ .$$

Recall that Baire's theorem asserts that, if E is a complete metric set, a countable intersection of open dense sets is dense. A set which contains such an intersection is called *generic*.

Proposition 4.5. *Let Ω be a Borel subset of $\mathbb{R}^d$ and $\partial\Omega^{ess}$ its essential boundary. The set $WB(\Omega)$ of weakly balanced points is generic in $\partial\Omega^{ess}$ (for the induced topology). As a consequence, the set of points $x_0 \in \partial\Omega^{ess}$ such that $\beta_w(x_0) = 0$ is generic in $\partial\Omega^{ess}$.*

Proposition 4.6. *Let Ω be a Borel subset of $\mathbb{R}^d$ and $\partial\Omega^{ess}$ its essential boundary. The set $SB(\Omega)$ of strongly balanced points is dense in $\partial\Omega^{ess}$. As a consequence, the set of points $x_0 \in \partial\Omega^{ess}$ such that $\beta_s(x_0) = 0$ is dense in $\partial\Omega^{ess}$.*

Remark: It would be interesting to determine if $SB(\Omega)$ is generic in $\partial\Omega^{ess}$.

Proof of Proposition 4.5. We first remark that Baire's theorem can be applied in $\partial\Omega^{ess}$ (because it is a closed subset of $\mathbb{R}^d$). Let $x_0 \in \partial\Omega^{ess}$ and $\varepsilon > 0$. Lebesgue's differentiability theorem, applied to the Borel function $f = \mathbb{1}_\Omega$ asserts that, for almost every $x \in \mathbb{R}^d$,

$$\frac{Vol(B(x,r) \cap \Omega)}{Vol(B(x,r))} \longrightarrow f(x) \quad \text{when} \quad r \longrightarrow 0 \ .$$

Recall that

$$Vol(\{x \in B(x_0, \varepsilon/2) \ ; \ f(x) = 1\}) > 0 \quad \text{and} \quad Vol(\{x \in B(x_0, \varepsilon/2) \ ; \ f(x) = 0\}) > 0 \ .$$

We can then find $y_0, \ y_1 \in B(x_0, \varepsilon/2)$ such that

$$\frac{Vol(B(y_0,r) \cap \Omega)}{Vol(B(y_0,r))} \geq \frac{3}{4} \quad \text{and} \quad \frac{Vol(B(y_1,r) \cap \Omega)}{Vol(B(y_1,r))} \leq \frac{1}{4}$$

when r is small enough.

Let $y_t = ty_1 + (1-t)y_0$. The intermediate value theorem applied to the continuous function

$$t \longmapsto \frac{Vol(B(y_t,r) \cap \Omega)}{Vol(B(y_t,r))}$$

allows us to construct a point $x_1 \in B(x_0, \varepsilon/2)$ (which is equal to y_t for some value of t) such that

$$\frac{Vol(B(x_1,r) \cap \Omega)}{Vol(B(x_1,r))} = \frac{1}{2} \ .$$

Such an open ball $B(x_1, r)$ will be called a "perfectly balanced" ball. The connexity of the ball $B(x_1, r)$ implies that it intersects $\partial\Omega^{ess}$ (remember that $\partial\Omega^{ess}$ is the topological boundary of the modified domain $\tilde{\Omega}$, see Proposition 4.3).

Let O_n be the union of all the open balls of radius $r \leq 1/n$ that are "perfectly balanced". We just have seen that $O_n \cap \partial\Omega^{ess}$ is an open dense subset of $\partial\Omega^{ess}$. So $\bigcap_{n\geq 1} O_n \cap \partial\Omega^{ess}$ is a countable intersection of open dense subsets of the essential boundary $\partial\Omega^{ess}$.

Moreover, if $x \in \bigcap_{n\geq 1} O_n \cap \partial\Omega^{ess}$, we can find a sequence of points $x_n \in \mathbb{R}^d$ and a sequence of positive real numbers $r_n \leq 1/n$ such that for every $n \geq 1$

$$x \in B(x_n, r_n) \quad \text{and} \quad Vol(B(x_n, r_n) \cap \Omega) = \frac{1}{2} Vol(B(x_n, r_n)) \ .$$

We then have

$$2^{-(d+1)} Vol(B(x, 2r_n)) \leq Vol(B(x, 2r_n) \cap \Omega) \leq (1 - 2^{-(d+1)}) Vol(B(x, 2r_n)),$$

which proves that $x \in WB(\Omega)$. •

Proof of Proposition 4.6. We develop the same idea as in Proposition 4.5. For commodity, we use the norm $\| \ \|_\infty$ instead of the euclidian norm in $\mathbb{R}^d$ and we will denote by $B_\infty(x,r)$ the "balls" related to this norm (which are cubes!). Let $x_0 \in \partial\Omega^{ess}$ and $\varepsilon > 0$. Using the same argument as in Proposition 4.5, we can find $x_1 \in B_\infty(x_0, \varepsilon/2)$ and $r \leq \varepsilon/2$ such that

$$\frac{Vol(\overline{B}_\infty(x_1,r) \cap \Omega)}{Vol(\overline{B}_\infty(x_1,r))} = \frac{1}{2} .$$

The closed cube $\overline{B}_\infty(x_1,r)$ can be divided into 2^d closed cubes of radius $r/2$ whose interiors do not overlap. Suppose that none of them is "perfectly balanced". We can then find two points $z_0,\ z_1$ such that

$$\begin{aligned} &\overline{B}_\infty(z_0,r/2) \subset \overline{B}_\infty(x_1,r), \qquad && Vol(\overline{B}_\infty(z_0,r/2) \cap \Omega) > \frac{1}{2} Vol(\overline{B}_\infty(z_0,r/2)) \\ &\overline{B}_\infty(z_1,r/2) \subset \overline{B}_\infty(x_1,r), \qquad && Vol(\overline{B}_\infty(z_1,r/2) \cap \Omega) < \frac{1}{2} Vol(\overline{B}_\infty(z_1,r/2)) . \end{aligned}$$

Using once again the intermediate value theorem, we can construct a point x_2 (which is a barycenter of z_0 and z_1) such that $\overline{B}_\infty(x_2,r/2) \subset \overline{B}_\infty(x_1,r)$ and such that the ball $\overline{B}_\infty(x_2,r/2)$ is "perfectly balanced". Iterating this construction we obtain a sequence of "perfectly balanced" cubes $\overline{B}_\infty(x_n, r2^{-(n-1)})$ such that

$$\overline{B}_\infty(x_{n+1}, r2^{-n}) \subset \overline{B}_\infty(x_n, r2^{-(n-1)}) .$$

Let $x_\infty = \lim_{n\to\infty} x_n$ and $0 < \rho \leq r$. Let us denote by n the integer such that

$$r2^{-n} < \frac{\rho}{\sqrt{d}} \leq r2^{-(n-1)} .$$

We observe that

$$B\left(x_\infty, \rho\right) \supset B_\infty\left(x_\infty, \rho/\sqrt{d}\right) \supset B_\infty\left(x_\infty, r2^{-n}\right) \supset B_\infty\left(x_{n+2}, r2^{-(n+1)}\right) .$$

In other words, the ball $B\left(x_\infty, \rho\right)$ contains a "perfectly balanced" cube with size length at least $\rho/2\sqrt{d}$. We deduce that

$$\begin{cases} Vol(B(x_\infty,\rho) \cap \Omega) \geq \dfrac{1}{2}\left(\dfrac{\rho}{2\sqrt{d}}\right)^d \\ Vol(B(x_\infty,\rho) \cap \Omega^c) \geq \dfrac{1}{2}\left(\dfrac{\rho}{2\sqrt{d}}\right)^d \end{cases} ; \tag{15}$$

(15) asserts that $x_\infty \in SB(\Omega)$. Moreover, $\|x_0 - x_\infty\|_\infty \leq \varepsilon$ and the proof is finished. •

4.3 The fractal dimension of the set of balanced points

We first consider the dimension of the set of points of accessibility 0.

Theorem 4.7. *Let Ω be a Borel subset of $\mathbb{R}^d$ such that $\partial\Omega^{ess} \neq \emptyset$. Then*

$$\dim_P (WB(\Omega)) \geq d - 1 \ .$$

Remark: In particular, $\dim_P(\partial\Omega^{ess}) \geq d - 1$.

Proof of Theorem 4.7. Let us begin with a lemma which is a slight modification of a well known result (see [6] or [9]).

Lemma 4.8. *Let G be a nonempty subset of $\mathbb{R}^d$ which satisfies Baire's property (for the induced topology) and $\delta > 0$. Suppose that for every $x \in G$, and every $r > 0$, $\Delta(G \cap B(x,r)) \geq \delta$. Then $\dim_P(G) \geq \delta$.*

Proof. Suppose that $G \subset \bigcup_{n\in\mathbb{N}} E_n$. Denote by $\overline{E}_n$ the closure (in $\mathbb{R}^d$) of E_n. Baire's property implies that one of the related closed sets $\overline{E}_n \cap G$ has an interior point in G. Thus there exist $x \in G$, $r > 0$ and $n_0 \in \mathbb{N}$ such that $G \cap B(x,r) \subset \overline{E}_{n_0} \cap G$, so that

$$\Delta(E_{n_0}) = \Delta(\overline{E}_{n_0}) \geq \Delta(\overline{E}_{n_0} \cap G) \geq \Delta(G \cap B(x,r)) \geq \delta$$

and Lemma 4.8 follows.

Proof of Theorem 4.7: As in Section 4.2, let O_n be the union of all "perfectly balanced" open cubes of radius $r \leq 1/n$ and let $G = \bigcap_{n\geq 1} O_n \cap \partial\Omega^{ess}$; G is a dense $\mathcal{G}_\delta$ of the Baire space $\partial\Omega^{ess}$, so that it satisfies Baire's property. Moreover, $G \subset WB(\Omega)$. According to Lemma 4.8, it is sufficient to prove that for every $x \in G$ and every $r > 0$, $\Delta(G \cap B(x,r)) \geq d - 1$. Let $x \in G$ and $r > 0$. We can find $y \in \mathbb{R}^d$ and $\rho > 0$ such that the cube $B_\infty(y,\rho)$ is "perfectly balanced" and $x \in B_\infty(y,\rho) \subset B(x,r)$. Let us split the cube $B_\infty(y,\rho)$ into 2^{dj} cubes of length $2^{-j+1}\rho$ which are called C_i. We want to estimate the number N_j of cubes C_i that intersect G. For each cube C_i, let $\omega(C_i) = Vol(C_i \cap \Omega)/Vol(C_i)$. The mean of $\omega(C_i)$ is $1/2$. So, at least $1/3^{th}$ of the $\omega(C_i)$ is greater than $1/4$ and $1/3^{th}$ of the $\omega(C_i)$ is lower than $3/4$. Now, there are two possibilities: Either $1/6^{th}$ of the cubes C_i are such that $1/4 \leq \omega(C_i) \leq 3/4$; all those cubes intersect G (see the proof of Proposition 4.5 and 4.6) and $N_j \geq 2^{dj}/6$. Else, there are at least $1/6^{th}$ of the cubes such that $\omega(C_i) \leq 1/4$ and $1/6^{th}$ of the cubes such that $\omega(C_i) \geq 3/4$. Let A be the union of all the closed cubes $\overline{C}_i$ such that $\omega(C_i) \leq 1/2$. Then

$$\frac{1}{6} Vol(B_\infty(y,\rho)) \leq Vol(A) \leq \frac{5}{6} Vol(B_\infty(y,\rho)) \ .$$

Isoperimetric inequalities (see for example [19]) ensure that the "surface" of the boundary of A is at least $C\rho^{d-1}$. In particular, there exist at least $C(\rho)2^{j(d-1)}$ couples of cubes (C, C') such that $\overline{C} \cap \overline{C'} \neq \emptyset$, $\omega(C) \leq 1/2$ and $\omega(C') \geq 1/2$. It follows that $\overline{C} \cap G \neq \emptyset$ or $\overline{C'} \cap G \neq \emptyset$ (an intermediate cube is "perfectly balanced"). It follows that $N_j \geq C2^{j(d-1)}$.

In either case, $N_j \geq C2^{j(d-1)}$. So, $\Delta(G \cap B(x,r)) \geq d - 1$. •

5 Multifractal properties of the essential boundary

5.1 Construction of the scaling function

We will construct a multifractal formalism based on the dyadic grid whose purpose is to derive the Hausdorff (or packing) dimensions of the level sets of the functions α_w and α_s. Recall that $\mathcal{F}_n$ is the set of dyadic (semi-open) cubes of scale n; denote by $\lambda_n(x)$ the unique cube in $\mathcal{F}_n$ that contains x. The following proposition is a simple consequence of the inclusions $B(x, 2^{-n}) \subset 3\lambda_n(x) \subset B(x, 3\sqrt{d}2^{-n})$.

Proposition 5.1. *Let Ω be a Borel subset of $\mathbb{R}^d$ and $x \in \partial\Omega^{ess}$. Then*

$$\alpha_w(x) + d = \liminf_{n\to+\infty} \frac{\log Vol\,(3\lambda_n(x) \cap \Omega)}{-n\log 2}, \quad \alpha_s(x) + d = \limsup_{n\to+\infty} \frac{\log Vol\,(3\lambda_n(x) \cap \Omega)}{-n\log 2} \ .$$

Proposition 5.1 suggests to introduce a scaling function as follows. Let Ω be a Borel set such that $\partial\Omega^{ess}$ is bounded and not empty; let

$$S(q,n) = \sum_{\lambda \in \mathcal{F}_n{}^*} (Vol(3\lambda \cap \Omega))^q \quad \text{where} \quad \mathcal{F}_n{}^* = \{\lambda \in \mathcal{F}_n : \ \ \lambda \cap \partial\Omega^{ess} \neq \emptyset\},$$

and

$$\tau(q) = \limsup_{n\to+\infty} \ \frac{1}{n\log 2} \log\left(S(q,n)\right) \ . \tag{16}$$

The function τ is decreasing and convex. The standard justification of the multifractal formalism runs as follows: First, the contribution to $S(q,n)$ of the set of points where the (weak or strong) accessibility exponent takes a given value α is estimated: If the dimension of this set is $d(\alpha)$, then there are about $2^{d(\alpha)n}$ dyadic cubes in $\mathcal{F}_n{}^*$ which cover this set; and such a cube satisfies $Vol(3\lambda \cap \Omega) \sim 2^{-\alpha n}$. Therefore the order of magnitude of the contribution we look for is $2^{-(\alpha q - d(\alpha))n}$. When $n \to +\infty$, the preponderent contribution is clearly obtained for the value of α that minimizes the exponent $\alpha q - d(\alpha)$; thus $\tau(q) = \inf_\alpha(\alpha q - d(\alpha))$. If $d(\alpha)$ is a concave function, then this formula can be inverted and $d(\alpha)$ is recovered from $\tau(q)$ by an inverse Legendre transform:

$$d(\alpha) = \inf_q(\alpha q + \tau(q)).$$

The multifractal formalism holds if, indeed, this relationship between the scaling function and the spectrum of singularties holds. We give in Section 5.3 some results in this direction.

Remark 1: The factor 3 in the definition of $S(q,n)$ is not always used in the derivation of the multifractal formalism for measures; however, it improves its range of validity, as shown by R. Riedi, see [18]. The novelty in our derivation is the restriction of the sum to the cubes λ such that $\lambda \cap \partial\Omega^{ess} \neq \emptyset$; this allows to eliminate all the points in

$\overset{\circ}{\Omega}^{ess}$ and in $\mathbb{R}^d \setminus \overline{\Omega}^{ess}$.

Remark 2: In [20], Testud already introduced such a "restricted" scaling function. In the context of his paper, a strange Cantor set K perturbs the multifractal analysis of the measure. Multifractal formalism breaks down at different levels. Testud introduces the scaling function τ_K in which the sum is restricted to the dyadic intervals that meet the Cantor set K and proves that for all the "bad exponents", the dimension of the level set is given by the Legendre transform τ_K^*.

5.2 Properties of the scaling function

Theorem 5.2. *Let Ω be a Borel subset of $\mathbb{R}^d$ such that $\partial\Omega^{ess}$ is nonempty and bounded. Define $\tau(q)$ as in* (16)*. The following properties hold.*

1. $\tau(0) = \Delta(\partial\Omega^{ess})$ *and* $\forall q \geq 0, \quad \tau(q) \leq \Delta(\partial\Omega^{ess}) - dq$.
2. $\forall q \geq 0, \quad \tau(q) \geq d - 1 - dq$.
3. $\forall q \in \mathbb{R}, \quad \dim_P (SB(\Omega)) \leq \tau(q) + dq$.
4. $\forall q \in \mathbb{R}, \quad \dim_H (WB(\Omega)) \leq \tau(q) + dq$.

Proof of Theorem 5.2.
1. If $\lambda \cap \partial\Omega^{ess} \neq \emptyset$, then, $Vol(3\lambda \cap \Omega) > 0$ and $(Vol(3\lambda \cap \Omega))^0 = 1$, thus $\tau(0) = \Delta(\partial\Omega^{ess})$. More precisely, if $q > 0$, then

$$\sum_{\lambda \in \mathcal{F}_n{}^*} (Vol(3\lambda \cap \Omega))^q \leq \text{Card}\,(\mathcal{F}_n{}^*)(3.2^{-n})^{dq};$$

it follows that $\tau(q) \leq \Delta(\partial\Omega^{ess}) - dq$.

2. If n is large enough, using a similar argument as in Theorem 4.7, we can find at least $c2^{(d-1)n}$ cubes in $\mathcal{F}_n{}^*$ which are "quite balanced". These cubes satisfy $Vol(3\lambda \cap \Omega) \sim 2^{-dn}$ and the inequality follows.

3. It is easy to see that $x_0 \in SB(\Omega)$ if and only if there exists $0 < \eta < 1/2$ and n_0 such that

$$\forall n \geq n_0, \quad \eta \leq \frac{Vol(3\lambda_n(x_0) \cap \Omega)}{(3.2^{-n})^d} \leq 1 - \eta \; . \tag{17}$$

Let $U_{n_0,\eta}$ denote the set of points that satisfy (17). Let $\alpha < \dim_P (SB(\Omega))$. We can find $p, n_0 \in \mathbb{N}^*$ such that

$$\Delta(U_{n_0,1/p}) \geq \dim_P (U_{n_0,1/p}) > \alpha \; .$$

If N_k is the number of cubes $\lambda \in \mathcal{F}_k$ we need to cover $U_{n_0,\,1/p}$, then, $N_k \geq 2^{k\alpha}$ infinitely often. Suppose $q > 0$ (the proof is similar if $q < 0$). We get

$$\sum_{\lambda \in \mathcal{F}_k{}^*} (Vol(3\lambda \cap \Omega))^q \geq N_k \left(\frac{1}{p} (3.2^{-k})^d \right)^q \geq \frac{3^{dq}}{p^q} 2^{k(\alpha - dq)}$$

infinitely often. We conclude that $\tau(q) \geq \alpha - dq$.

4. Note that $x_0 \in WB(\Omega)$ if and only if there exists $0 < \eta < 1/2$ such that

$$\forall n_0,\ \exists n \geq n_0\ ;\quad \eta \leq \frac{Vol(3\lambda_n(x_0) \cap \Omega)}{(3.2^{-n})^d} \leq 1 - \eta\ . \tag{18}$$

Let V_η denote the set of points that satisfy (18). Let $p \in \{2, 3, \cdots\}$, $n_0 \in \mathbb{N}^*$ and suppose that $q > 0$ (the proof is similar if $q < 0$). We can cover $V_{1/p}$ with cubes of scale $n \geq n_0$ such that $Vol(3\lambda \cap \Omega) \geq \frac{1}{p}(3.2^{-n})^d$. Let $\mathcal{R}$ be such a covering and $\tau' > \tau(q)$. We have

$$\begin{aligned} \sum_{\lambda \in \mathcal{R}} \mathrm{diam}\,(\lambda)^{\tau' + dq} &\leq C \sum_{\lambda \in \mathcal{R}} (Vol(3\lambda \cap \Omega))^q \ \mathrm{diam}\,(\lambda)^{\tau'} \\ &\leq C \sum_{n \geq n_0} \left[\sum_{\lambda \in \mathcal{F}_n{}^*} (Vol(3\lambda \cap \Omega))^q \right] 2^{-n\tau'}\ . \end{aligned}$$

Moreover, if $\tau' > \tau'' > \tau(q)$ and n_0 sufficiently large, then $\sum_{\lambda \in \mathcal{F}_n{}^*} (Vol(3\lambda \cap \Omega))^q \leq 2^{n\tau''}$. It follows that

$$\sum_{\lambda \in \mathcal{R}} \mathrm{diam}\,(\lambda)^{\tau' + dq} \leq C \sum_{n \geq n_0} 2^{n(\tau'' - \tau')} \leq \frac{C}{1 - 2^{\tau'' - \tau'}}\ .$$

We conclude that $\dim_H (V_{1/p}) \leq \tau' + dq$ and $\dim_H (WB(\Omega)) \leq \tau' + dq$.

5.3 The multifractal formalism associated with $\partial\Omega^{ess}$

The proofs of points 3 and 4 in Theorem 5.2 allow to obtain estimates of the level sets of accessibility index.

Theorem 5.3. *Let Ω be a Borel subset of $\mathbb{R}^d$ such that $\partial\Omega^{ess}$ is nonempty and bounded. Define $\tau(q)$ as in* (16)*. If $\alpha \geq 0$, let*

$$E^w_\alpha = \{x \in \partial\Omega^{ess}\ ;\ \alpha_w(x) \leq \alpha\} \quad \textit{and} \quad E^s_\alpha = \{x \in \partial\Omega^{ess}\ ;\ \alpha_s(x) \leq \alpha\}\ .$$

For every $q > 0$,

$$\dim_H (E^w_\alpha) \leq (d + \alpha)q + \tau(q) \quad \textit{and} \quad \dim_P (E^s_\alpha) \leq (d + \alpha)q + \tau(q)\ .$$

In particular, if $\alpha + d \leq -\tau'_-(0)$, then

$$\dim_H (E^w_\alpha) \leq \tau^*(\alpha + d) \quad \textit{and} \quad \dim_P (E^s_\alpha) \leq \tau^*(\alpha + d)\ .$$

The proof uses the same ideas as in Theorem 5.2 and requires to introduce the set of points $x \in \partial\Omega^{ess}$ such that $Vol\,(3\lambda_n(x) \cap \Omega) \geq 2^{-n(\alpha+d+\varepsilon)}$ infinitely often (resp. for n large enough). In the same way, we can also prove the following twin result.

Theorem 5.4. *Let Ω be a Borel subset of $\mathbb{R}^d$ such that $\partial\Omega^{ess}$ is nonempty and bounded. Define $\tau(q)$ as in (16). If $\alpha \geq 0$, let*

$$F_\alpha^w = \{x \in \partial\Omega^{ess}\ ;\ \alpha_w(x) \geq \alpha\} \quad \textit{and} \quad F_\alpha^s = \{x \in \partial\Omega^{ess}\ ;\ \alpha_s(x) \geq \alpha\}\ .$$

For every $q < 0$,

$$\dim_P(F_\alpha^w) \leq (d+\alpha)q + \tau(q) \quad \textit{and} \quad \dim_H(F_\alpha^s) \leq (d+\alpha)q + \tau(q)\ .$$

In particular, if $\alpha + d \geq -\tau'_+(0)$,

$$\dim_P(F_\alpha^w) \leq \tau^*(\alpha+d) \quad \textit{and} \quad \dim_H(F_\alpha^s) \leq \tau^*(\alpha+d)\ .$$

Remark 1: The set E_α^s (resp. F_α^w) is quite similar to the set of strong α-accessible points (resp. weak α-accessible points).

Remark 2: The results in Theorem 5.3 and 5.4 are standard multifractal inequalities adapted to the context of boundaries (see [3]).

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